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Helicon waves in non-resistive cylindrical and spherical plasmas

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Abstract. A theoretical discussion of helicon wave propagation in non-resistive plasmas is presented. The boundary conditions are derived and applied in cylindrical geometry to give results consistent with experiment. The problem is more complicated in spherical geometry because of the non-separability of the wave equation. However, by separating in cylindrical coordinates and rearranging in a series of spherical waves, solutions are obtained in the form of an infinite set of integral equations. Approximate solutions are obtained and the first few natural frequencies are identified. Wave fields for the principal mode are calculated and displayed graphically.

1. Introduction

A helicon wave is a low-frequency magnetic disturbance which propagates in a highly conducting medium, such as a metal at very low temperatures or in a gas discharge plasma, when a strong magnetic field is applied. The macroscopic theory of such waves has been studied by Aigrain (1961), Bowers *et al.* (1961), Legendy (1964) and others. The cylindrical plasma has been treated very thoroughly by Klozenberg *et al.* (1965).

It has been suggested by Aigrain (1964) that the spherical plasma would be most convenient for studying interactions of helicon waves with other waves, such as acoustic waves, in solid state plasmas. This is because the sample may be rotated so that the applied magnetic field makes different angles with the various crystal planes without altering the boundary conditions on the sample. The alternative of using a new sample for the study of each crystal plane presents the practical difficulty of making consistent samples.

In this paper the plasma is assumed to have zero resistivity. This requires the introduction of a new set of boundary conditions which are first applied to a cylindrical plasma for verification. The dispersion relation which results differs slightly from that of Klozenberg *et al.* (1965) but agrees well with experiment.

For the non-resistive spherical plasma, the equation

$$\boldsymbol{B} \cdot \nabla \nabla \times \boldsymbol{b} = \mathrm{i} a \boldsymbol{b} \tag{1.1}$$

where $\boldsymbol{B}$ is the applied uniform magnetic field, $\boldsymbol{b}$ is a magnetic-field perturbation and a is a real positive constant, must be solved in spherical coordinates and the solution matched with an external magnetic field for which $\nabla \times \boldsymbol{b} = 0$. For problems which have been solved previously it is possible to assume solutions of the form

$$\boldsymbol{b} = \boldsymbol{b}'(x, y) \exp\{\mathrm{i}(kz - \omega t)\}$$

that is a wave propagating without change of shape along the direction of the field $\boldsymbol{B} = B\hat{\boldsymbol{z}}$. This assumption may not be made here and it is not obvious that (1.1) has wave-like solutions in a sphere. In the paper it is shown that solutions exist which resemble an infinite series of superimposed spherical waves. The resonance condition consists of an infinite set of simultaneous equations which do not appear to belong to any known classification. An approximate solution is obtained which yields a set of resonant frequencies and the corresponding fields.

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2. The basic assumptions and equations

The macroscopic theory of helicon waves may be based on the equation

$$\boldsymbol{E} - \frac{1}{Ne}\boldsymbol{j} \times \boldsymbol{B}_{\mathrm{T}} = \eta \boldsymbol{j} \tag{2.1}$$

where $\boldsymbol{E}$ is the total electric field, $\boldsymbol{B}_{\mathrm{T}}$ is the total magnetic field, $\boldsymbol{j}$ is the current density, N is the electron number density and η is the electrical resistivity. The quantity $-1/Ne$ is the Hall coefficient. Equation (2.1) can be derived from the modified Ohm's law (Klozenberg *et al.* 1965) under the following assumptions:

(i) The plasma is cool, so that electron pressure can be neglected.

(ii) Ion motion is negligible.

(iii) The plasma is collision dominated, that is the collision frequency is much greater than the frequency of the waves under consideration, thus $\mathrm{d}j/\mathrm{d}t$ does not appear in (2.1).

Suppose the current $\boldsymbol{j}$ and the electric field $\boldsymbol{E}$ arise from a perturbation $\boldsymbol{b}(\boldsymbol{r}, t)$ in a large steady uniform magnetic field $\boldsymbol{B}$, so that

$$\boldsymbol{B}_{\mathrm{T}}(\boldsymbol{r}, t) = \boldsymbol{B} + \boldsymbol{b}(\boldsymbol{r}, t) \tag{2.2}$$

where $|\boldsymbol{B}| \gg |\boldsymbol{b}|$. Inserting (2.2) in (2.1) and linearizing gives

$$\boldsymbol{E} - \frac{1}{Ne}\boldsymbol{j} \times \boldsymbol{B} = \eta \boldsymbol{j} \tag{2.3}$$

which is to be solved in conjunction with Maxwell's equations for the perturbed quantities:

$$\nabla \times \boldsymbol{b} = \mu_0 \boldsymbol{j} \tag{2.4}$$

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{b}}{\partial t} \tag{2.5}$$

$$\nabla \cdot \boldsymbol{b} = 0 \tag{2.6}$$

where the further assumption, (iv) that displacement current $\partial \boldsymbol{D}/\partial t$ is negligible, has been made. That is

$$\left|\frac{\partial \boldsymbol{D}}{\partial t}\right| \ll |\boldsymbol{j}|. \tag{2.7}$$

It will be shown below (see (3.4)) that, for a highly conducting plasma, (2.3) reduces to

$$\boldsymbol{j} = \frac{1}{\eta}\hat{\boldsymbol{B}}\hat{\boldsymbol{B}} \cdot \boldsymbol{E} \qquad \left(\hat{\boldsymbol{B}} = \frac{\boldsymbol{B}}{B}\right).$$

Thus for a disturbance of frequency ω (2.7) implies that

$$\omega|\boldsymbol{D}| \ll \frac{1}{\eta}|\boldsymbol{E}|.$$

Hence

$$\omega \ll \frac{1}{\eta\epsilon_0}.$$

For a sodium plasma at 4 °K, $1/\eta\epsilon_0 \sim 10^{22}$, so that the condition certainly holds. It is clear that this condition always holds in the limit of zero resistivity.

From (2.3), (2.4) and (2.5) there results, in the limit $\eta \to 0$, the basic wave equation for the field $\boldsymbol{b}$ in the plasma:

$$\frac{\partial \boldsymbol{b}}{\partial t} + \frac{1}{Ne\mu_0}\boldsymbol{B} \cdot \nabla\nabla \times \boldsymbol{b} = 0. \tag{2.8}$$

3. Boundary conditions

The boundary conditions in the zero-resistance limit are deduced from the equations of finite conductivity (2.3) to (2.6). Integrating (2.4) across the plasma–vacuum interface gives for the jump $[\boldsymbol{b}]$ in $\boldsymbol{b}$ on crossing the boundary

$$\boldsymbol{n} \times [\boldsymbol{b}] = \mu_0 \boldsymbol{K} \tag{3.1}$$

where $\boldsymbol{n}$ is the unit normal to the boundary and $\boldsymbol{K}$ is the surface current density.

Now $\boldsymbol{K} = \lim h\boldsymbol{j}$ as $h \to 0$, where h is a coordinate normal to the surface. For finite conductivity (2.3) shows that $\boldsymbol{j}$ is finite and hence $\boldsymbol{K} = 0$. If the conductivity is allowed to become infinite, then $\boldsymbol{j}$ may be infinite and $\boldsymbol{K}$ non-zero as follows. Solving (2.3) for $\boldsymbol{j}$ gives

$$\boldsymbol{j} = \frac{1}{\eta(1+\xi^2)}\{\hat{\boldsymbol{B}}(\hat{\boldsymbol{B}} \cdot \boldsymbol{E}) + \xi\hat{\boldsymbol{B}} \times \boldsymbol{E} + \xi^2 \boldsymbol{E}\} \tag{3.2}$$

where ξ is the dimensionless quantity $Ne\eta/B$. Suppose that $\xi \ll 1$, then

$$\eta \ll \frac{B}{Ne} \tag{3.3}$$

and

$$\boldsymbol{j} \simeq \frac{1}{\eta}\hat{\boldsymbol{B}}(\hat{\boldsymbol{B}} \cdot \boldsymbol{E}). \tag{3.4}$$

(In sodium at 4 °K, $\xi \simeq 0{\cdot}02$.) At the plasma surface the current has a tangential component

$$\boldsymbol{j}_{\mathrm{T}} = \frac{1}{\eta}(\hat{\boldsymbol{B}} \cdot \boldsymbol{E})(\hat{\boldsymbol{B}} \cdot \boldsymbol{t})\boldsymbol{t}$$

where $\boldsymbol{t}$ is the unit tangent vector in the plane containing $\boldsymbol{n}$ and $\boldsymbol{B}$. Now suppose that $1/\eta$ becomes infinite in such a way that the 'surface conductivity' $\Sigma = h/\eta$ remains finite. Then

$$\begin{aligned} \boldsymbol{K} &= \lim_{h \to 0} h\boldsymbol{j}_{\mathrm{T}} \\ &= \lim_{h \to 0} \frac{h}{\eta}(\hat{\boldsymbol{B}} \cdot \boldsymbol{E})(\hat{\boldsymbol{B}} \cdot \boldsymbol{t})\boldsymbol{t} \\ &= (\Sigma\hat{\boldsymbol{B}} \cdot \boldsymbol{E}\,\hat{\boldsymbol{B}} \cdot \boldsymbol{t})\boldsymbol{t}. \end{aligned}$$

From (3.1)

$$\boldsymbol{n} \times [\boldsymbol{b}] = \mu\Sigma(\hat{\boldsymbol{B}} \cdot \boldsymbol{E})(\hat{\boldsymbol{B}} \cdot \boldsymbol{t})\boldsymbol{t} \tag{3.5}$$

and from (2.6)

$$\boldsymbol{n} \cdot [\boldsymbol{b}] = 0. \tag{3.6}$$

Expressions (3.5) and (3.6) are the boundary conditions for the field $\boldsymbol{b}$. However, only two boundary conditions are required to solve the boundary value problem, and these can be chosen to be independent of the surface current since, from (3.5),

$$\boldsymbol{t} \times (\boldsymbol{n} \times [\boldsymbol{b}]) = 0$$

then

and

$$\left.\begin{aligned} \boldsymbol{t} \cdot [\boldsymbol{b}] &= 0 \\ \boldsymbol{n} \cdot [\boldsymbol{b}] &= 0 \end{aligned}\right\}. \tag{3.7}$$

An alternative expression of these conditions is

$$(\boldsymbol{n} \times \hat{\boldsymbol{B}}) \times [\boldsymbol{b}] = 0. \tag{3.8}$$

The boundary conditions for helicons have been the subject of some controversy (Legendy 1964, p. A1716). Formula (3.7) is a new set of boundary conditions, the consequences of which are compared with some established results in the next section.

The surface current mentioned above is not of great physical interest since it would be difficult to measure, in sodium for example. However, for a cylindrical boundary which is parallel to $\boldsymbol{B}$ at every point $\boldsymbol{t} = \hat{\boldsymbol{B}}$. Therefore

$$\boldsymbol{K} = \Sigma(\hat{\boldsymbol{B}} \,.\, \boldsymbol{E})\hat{\boldsymbol{B}}.$$

But from (2.3) $\boldsymbol{E} = (1/Ne)\boldsymbol{j} \times \boldsymbol{B}$ when $\eta = 0$. Therefore $\boldsymbol{K} = 0$ at every point of the boundary.

4. Helicon waves in a non-resistive cylindrical plasma

If we choose the z coordinate to be parallel to the uniform field $\boldsymbol{B}$, equation (2.8) reduces to

$$\frac{\partial \boldsymbol{b}}{\partial t} + \frac{B}{Ne\mu_0}\frac{\partial}{\partial z}\nabla \times \boldsymbol{b} = 0. \tag{4.1}$$

Solutions of the form

$$\boldsymbol{b}(\rho, z, t) = \hat{\boldsymbol{b}}(\rho)\exp\{\mathrm{i}(kz - \omega t)\} \tag{4.2}$$

are sought. This gives

$$\nabla \times \boldsymbol{b} = \alpha \boldsymbol{b} \tag{4.3}$$

where $\alpha = \omega Ne\mu_0/Bk$ and $\nabla . \boldsymbol{b} = 0$ gives

$$\nabla^2 \boldsymbol{b} = \alpha^2 \boldsymbol{b}. \tag{4.4}$$

The solutions for the cylindrical components are

$$\hat{b}_z = A\mathrm{J}_0(\gamma\rho)$$

where J_0 is the Bessel function of order zero and

$$\gamma^2 = \alpha^2 - k^2$$

$$\hat{b}_\rho = -\frac{\mathrm{i}kA}{\gamma}\mathrm{J}_1(\gamma\rho)$$

and

$$\hat{b}_\phi = \frac{\alpha A}{\gamma}\mathrm{J}_1(\gamma\rho).$$

Outside the cylinder there is no current, so that $\boldsymbol{b} = \nabla\psi$, where ψ is a scalar and $\nabla^2\psi = 0$. The appropriate solution having the form of (4.2) is $\hat{\psi} = C\mathrm{K}_0(k\rho)$, where K_0 is the modified Bessel function of order zero and C is a constant. The external component solutions are then

$$\hat{b}_\rho = Ck\mathrm{K}_1(k\rho)$$
$$\hat{b}_z = \mathrm{i}kC\mathrm{K}_0(k\rho)$$
$$\hat{b}_\phi = 0.$$

The boundary conditions (3.7) imply here that b_ρ and b_z are continuous at the surface $\rho = \rho_0$ of the cylinder. Therefore

$$\frac{\mathrm{i}kA}{\gamma}\mathrm{J}_1(\gamma\rho_0) = Ck\mathrm{K}_1(k\rho_0)$$

and

$$A\mathrm{J}_0(\gamma\rho_0) = \mathrm{i}kC\mathrm{K}_0(k\rho_0).$$

These are linear equations in A and C, and for consistency

$$\frac{\gamma\mathrm{J}_0(\gamma\rho_0)}{\mathrm{J}_1(\gamma\rho_0)} + \frac{k\mathrm{K}_0(k\rho_0)}{\mathrm{K}_1(k\rho_0)} = 0. \tag{4.5}$$

Equation (4.5) shows that ω is a real multi-valued function of k. That is, $\omega_n = f_n(k)$ for $n = 1, 2, \ldots$. These functions define a set of modes and their dispersion relations. The functions (4.5) have been computed and shown to compare well with experimental measurements on sodium and indium (Hui 1966).

The dispersion relation (4.5) differs from that of Klozenberg *et al.* (1965, equation (3.2)) by the purely imaginary term $\mathrm{i}\alpha$, where

$$\alpha = \frac{Ne\omega\mu_0}{Bk} = \frac{\mu_0}{k\eta}\xi.$$

This term diminishes with decreasing wavelength since $\alpha \to 0$ as $k \to \infty$.

The discontinuity in $\hat{b}_\phi$ given here as

$$[\hat{b}_\phi] = -\frac{A\alpha}{\gamma}\mathrm{J}_1(\gamma\alpha)$$

is the same as that obtained by Klozenberg *et al.* (1965, equation (4.2)). A discontinuity in $\hat{b}_z$ in the limit $\eta \to 0$, representing a non-zero surface current, which is obtained by Klozenberg does not appear in the present approximation.

5. Solutions in a spherical plasma

Monochromatic axially symmetric solutions $\boldsymbol{b}(\boldsymbol{r}, t) = \boldsymbol{b}'(\boldsymbol{r})\exp(\mathrm{i}\omega t)$ are sought for which the wave equation is

$$\mathrm{i}C^2\boldsymbol{b} = \frac{\partial}{\partial z}\nabla \times \boldsymbol{b} \tag{5.1}$$

where $C^2 = \omega Ne\mu_0/B$, and $\boldsymbol{B} = B\hat{\boldsymbol{z}}$. To satisfy boundary conditions on a sphere, (5.1) has to be solved in spherical coordinates. The normal straightforward procedure yields a fourth-order partial differential equation for one of the spherical components of $\boldsymbol{b}$. This equation does not appear to separate nor does it seem to be susceptible to any standard method of solution.

We now show that it is possible to find an integral representation of the spherical solutions in terms of the solution in cylindrical coordinates (ρ, ϕ, z). The cylindrical component equations are

$$\mathrm{i}C^2 b_\rho = -\frac{\partial^2 b_\phi}{\partial z^2}$$

$$\mathrm{i}C^2 b_\phi = \frac{\partial}{\partial z}\left(\frac{\partial b_\rho}{\partial z} - \frac{\partial b_z}{\partial \rho}\right)$$

$$\mathrm{i}C^2 b_z = \frac{\partial}{\partial z}\left\{\frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho b_\phi)\right\}.$$

From these a separable equation is obtained for b_ϕ with the solution

$$b_\phi = \mathrm{J}_1(\lambda\rho)\{A\exp(\mathrm{i}hz) + B\exp(-\mathrm{i}hz) + C\exp(qz) + D\exp(-qz)\} \tag{5.2}$$

where

$$h = \frac{1}{\sqrt{2}}\{(\lambda^4 + 4C^4)^{1/2} - \lambda^2\}^{1/2} \quad \text{and} \quad q = \frac{1}{\sqrt{2}}\{(\lambda^4 + 4C^4)^{1/2} + \lambda^2\}^{1/2}$$

and

$$\lambda^2 = \frac{C^4 - h^4}{h^2}.$$

A simple product solution such as this will only satisfy boundary conditions on a cylindrical boundary. To satisfy spherical boundary conditions, the general solution, which is a sum or integral over all allowed values of h and q, is needed. The required solutions, which in addition are wave-like, are

$$b_\phi = \int_{-C}^{C} \mathrm{d}h\, A(h) \exp(\mathrm{i}hz) \mathrm{J}_1(\lambda\rho) \tag{5.3}$$

where $A(h)$ is an arbitrary function of h.

This result may be expressed in a form which separates the r and θ dependence by using an identity found in the work of Stratton (1941, p. 413), so that we have with

$$\rho = r\cos\theta,\ z = r\sin\theta$$

$$b_\phi = \sum_{n=1}^{\infty} \mathrm{i}^{n-1}(2n+1)\frac{(n-1)!}{(n+1)!}\int_{-C}^{C} \mathrm{d}h\, A(h)\mathrm{P}_n^{\,m}\left(\frac{h^2}{C^2}\right)\mathrm{j}_n\left(\frac{C^2}{h}r\right)\mathrm{P}_n^{\,1}(\cos\theta) \tag{5.4}$$

where the $\mathrm{P}_n^{\,m}$ are associated Legendre functions and the j_n are spherical Bessel functions. By using (5.1) in spherical polar coordinates we obtain

$$b_r = \sum_{n=1}^{\infty} \mathrm{i}^{n-1}(2n+1)\left\{\frac{1}{r}\int_{-C}^{C} \mathrm{d}h\, A(h)\frac{h}{C^2}\mathrm{P}_n^{\,1}\left(\frac{h^2}{C^2}\right)\mathrm{j}_n\left(\frac{C^2}{h}r\right)\mathrm{P}_n^{\,1}(\cos\theta)\right\} \tag{5.5}$$

and

$$b_\theta = \sum_{n=1}^{\infty} \mathrm{i}^{n-1}(2n+1)\frac{(n-1)!}{(n+1)!}\left[\frac{1}{r}\int_{-C}^{C} \mathrm{d}h\, A(h)\frac{h}{C^2}\mathrm{P}_n^{\,1}\left(\frac{h^2}{C^2}\right)\left\{n\mathrm{j}_n\left(\frac{C^2}{h}r\right) - \frac{C^2 r}{h}\mathrm{j}_{n-1}\left(\frac{C^2}{h}r\right)\right\}\right] \times \mathrm{P}_n^{\,1}(\cos\theta). \tag{5.6}$$

Outside the spherical plasma we have $\nabla\times\boldsymbol{b} = 0$, and setting $\boldsymbol{b} = \nabla\psi$ we have $\nabla^2\psi = 0$, so that the spherical components of $\boldsymbol{b}$ are

$$b_r = -\sum_n B_n(n+1)r^{-(n+2)}\mathrm{P}_n(\cos\theta) \tag{5.7}$$

$$b_\theta = -\sum_n B_n r^{-(n+2)}\mathrm{P}_n^{\,1}(\cos\theta) \tag{5.8}$$

$$b_\phi = 0.$$

The boundary conditions are that b_r and b_θ are continuous at the surface $r = a$ of the sphere. These two conditions are sufficient to determine the arbitrary constants B_n and the arbitrary function $A(h)$. Equating coefficients of $\mathrm{P}_n(\cos\theta)$ in (5.5) and (5.7) at $r = a$ we obtain

$$\mathrm{i}^{n-1}(2n+1)\left\{\frac{1}{a}\int_{-C}^{C} \mathrm{d}h\, A(h)\frac{h}{C^2}\mathrm{P}_n^{\,1}\left(\frac{h^2}{C^2}\right)\mathrm{j}_n\left(\frac{C^2 a}{h}\right)\right\} = -B_n(n+1)a^{-(n+2)} \tag{5.9}$$

while (5.6) and (5.8) give

$$\mathrm{i}^{n-1}\frac{2n+1}{n(n+1)}\left[\frac{1}{a}\int_{-C}^{C} \mathrm{d}h\, A(h)\frac{h}{C^2}\mathrm{P}_n^{\,1}\left(\frac{h^2}{C^2}\right)\left\{n\mathrm{j}_n\left(\frac{C^2 a}{h}\right) - \frac{C^2 a}{h}\mathrm{j}_{n-1}\left(\frac{C^2 a}{h}\right)\right\}\right] = -B_n a^{-(n+2)}. \tag{5.10}$$

The requirement of consistency between (5.9) and (5.10) gives

$$\int_{-1}^{1} \mathrm{d}x\, f(x)\mathrm{P}_n^{\,1}(x^2)\mathrm{j}_{n-1}\left(\frac{y}{x}\right) = 0 \tag{5.11}$$

where $x = h/C$, $f(x) = CA(Cx)$, $y = Ca$ and n has the values 1, 2, 3, ..., ∞. The set of constants y_i and the corresponding functions $f_i(x)$, if they exist, define a set of resonant frequencies $\omega_i = \omega_0 y_i^2$ and corresponding wave fields $\boldsymbol{b}_i$.

Unfortunately, it has not been possible to solve (5.11) nor to establish whether, and under what conditions, solutions of the required form exist. To proceed, we assume that solutions exist, and in the next section solutions are obtained after an approximation has been made.

6. Approximate solutions

Equation (5.11) is in fact a set of simultaneous equations since the boundary conditions must be satisfied simultaneously for all values of n. Each term of the series (5.5) and (5.6) is not separately a solution of the helicon wave equation, although each term of the series (5.7) and (5.8) is separately a solution of Laplace's equation. This is in contrast with the problem of resonance of electromagnetic waves in a dielectric sphere (Stratton 1941, p. 554), where it is possible to equate a single term of the internal series with a term of the external series, while setting the arbitrary constants (the B_n) in the other terms equal to zero. Here the B_n can be determined only after (5.11) has been solved.

The limits of integration in (5.11) may be changed to 0, 1 using the symmetry property $\mathrm{j}_n(-z) = (-1)^n \mathrm{j}_n(z)$. We observe that the major contribution to the integral by the function $\mathrm{j}_n(y/x)$ comes from the region near $x = 1$, which suggests a series expansion about this point. As a first approximation we replace $\mathrm{j}_n(y/x)$ by its value $\mathrm{j}_n(y)$ at $x = 1$. This is apparently a drastic assumption, but we have calculated improved solutions using a perturbation method (see appendix) and found an error of about 20% in the first approximation to the eigenfrequencies. We now have in place of (5.11) a set of integrals of the form

$$\mathrm{j}_{2n-1}(y) \int_0^1 \mathrm{d}x\, \mathrm{P}_{2n}^1(x^2)\{f(x) - f(-x)\} = 0$$

$$\mathrm{j}_{2n-2}(y) \int_0^1 \mathrm{d}x\, \mathrm{P}_{2n-1}^1(x^2)\{f(x) + f(-x)\} = 0$$

which may be solved exactly for the solutions $f(x) = D_{2k} x P_{2k}^1(x^2)$, corresponding to $y = x_{2k-1,s}$, when $f(x)$ is an even function, and $f(x) = -\mathrm{i}D_{2k-1}|x|\mathrm{P}_{2k-1}^1(x^2)$, corresponding to $y = x_{2k-2,s}$, when $f(x)$ is odd. Here the x_m are zeros of $\mathrm{j}_m(y)$ and the D_m are arbitrary constants. These solutions enable the external and internal fields to be calculated.

The natural (resonant) frequencies $\omega_{n,s}$ are given by

$$\frac{\omega_{n,s}}{\omega_0} = X_{n,s}^2$$

where $\omega_0 = B/Ne\mu_0 a^2$ and $X_{n,s}$ are zeros of $\mathrm{j}_{n-1}(y)$. Values of ω/ω_0 have been calculated and are given in table 1 for $n = 1$ to 5 and $s = 1$ to 5.

Table 1

s \ n	1	2	3	4	5
1	9·9	20·2	33·1	48·9	66·9
2	39·4	59·8	82·9	109	137
3	89	119	152	188	226
4	158	198	241	286	335
5	247	297	349	405	464

The wave fields have been calculated for the mode of lowest frequency, and these are given approximately by

$$b_{\phi,1}\ (\text{internal}) = D_1(0{\cdot}625\mathrm{j}_2\mathrm{P}_2{}^1 + 0{\cdot}188\mathrm{j}_4\mathrm{P}_4{}^1 + 0{\cdot}102\mathrm{j}_6\mathrm{P}_6{}^1)$$

$$b_{r,1}\ (\text{internal}) = -\frac{D_1\mathrm{i}}{u}(0{\cdot}359\mathrm{j}_1\mathrm{P}_1 - 0{\cdot}570\mathrm{j}_3\mathrm{P}_3 - 0{\cdot}249\mathrm{j}_5\mathrm{P}_5)$$

$$b_{\theta,1}\ (\text{internal}) = -\mathrm{i}D_1\left\{\frac{1}{u}(0{\cdot}180\mathrm{j}_1\mathrm{P}_1{}^1 - 0{\cdot}143\mathrm{j}_3\mathrm{P}_3{}^1 - 0{\cdot}42\mathrm{j}_5\mathrm{P}_5) - \mathrm{j}_0\mathrm{P}_1{}^1\right\}$$

$$b_{\phi,1}\ (\text{external}) = 0$$

$$b_{r,1}\ (\text{external}) = -\mathrm{i}D_1(0{\cdot}114u^{-3}\mathrm{P}_1 - 0{\cdot}095u^{-5}\mathrm{P}_3 - 0{\cdot}005u^{-7}\mathrm{P}_5)$$

$$b_{\theta,1}\ (\text{external}) = -\mathrm{i}D_1(0{\cdot}057u^{-3}\mathrm{P}_1{}^1 - 0{\cdot}023u^{-5}\mathrm{P}_3{}^1 - 0{\cdot}0008u^{-7}\mathrm{P}_5{}^1).$$

Here $u = r/a$, the j_n have the argument πu and the P_n have the argument $\cos\theta$.

These functions of u and θ have been plotted in a number of ways to display the field configuration. By calculating the components $b_{r,1}$ and $b_{\theta,1}$ at a number of points in an axial plane we obtain a rough plot of the field lines. This is shown as figure 1. It can be

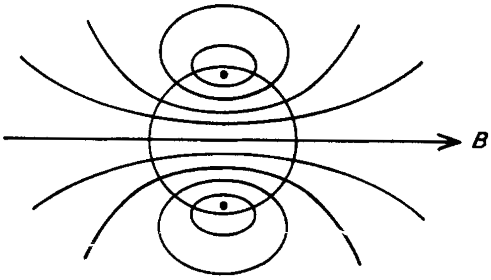

Figure 1. Approximate field line configuration in the sphere.

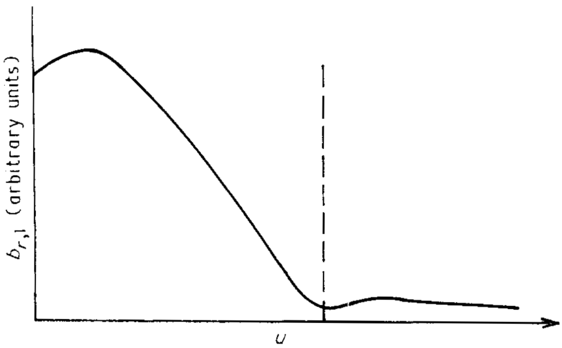

Figure 2. Field on the axis of the sphere as a function of $u = r/a$.

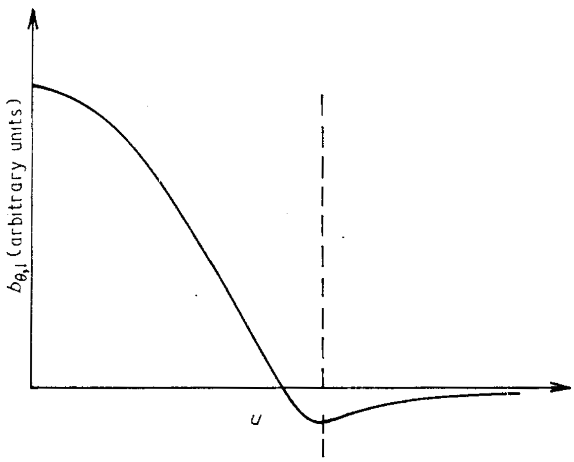

Figure 3. Field in the equatorial plane as a function of u.

seen that the field resembles that of a circular current loop in an equatorial plane with radius approximately that of the spherical plasma.

On the axis of the sphere $\theta = 0$, so that $P_n^1(1) = 0$ and $P_n(1) = 1$. This gives $b_{\phi,1} = b_{\theta,1} = 0$. Figure 2 shows $b_{r,1}$ as a function of u for this case.

On the equatorial plane $\theta = \frac{1}{2}\pi$, so that $\cos\theta = 0$, $P_n^1(0) = 0$, giving $b_{\phi,1} = 0$. Likewise $P_{2n-1}(0) = 0$, so that $b_{r,1} = 0$. Figure 3 is a plot of $b_{\theta,1}$ as a function of u for this case.

On the surface $u = 1$, and here we plot all three components as functions of θ. This is figure 4.

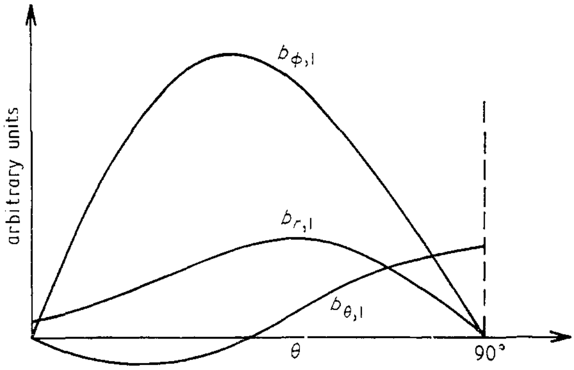

Figure 4. Field at the surface of the sphere.

7. Discussion

The preceding theory predicts the existence of natural modes of oscillation of helicon waves in a spherical plasma. The frequencies of the lower modes have been calculated, and it should be possible to excite these experimentally and so measure their frequencies.

The boundary conditions have been deduced from first principles and appear to conform to experimental measurements in a cylindrical plasma, so that they may be used in any bounded conductor of sufficiently low resistivity.

The wave fields resemble an infinite series of superimposed spherical waves of different wavelength. The types of mode and their degeneracy are similar to those obtained for oscillations of electromagnetic waves on a dielectric sphere. Solutions having axial symmetry have been considered here, but the theory can be extended to deal with axial dependence. These solutions are entirely degenerate, however; they merely add more modes of the same frequency.

The problem has a number of interesting mathematical features. The helicon wave equation in a sphere separates in cylindrical coordinates but not in spherical coordinates. This may be related to the fact that the wave equation is invariant under rotation about $\boldsymbol{B}$ but not about any other direction. The conventional wave equation is spherically symmetric and separable in both coordinate systems.

The method of obtaining solutions has relied on a number of special relations between the coordinate systems and the corresponding solutions.

The coordinate systems have the component b_ϕ in common. The solution (5.3) for b_ϕ has the same form as a conventional cylindrical wave, apart from the dependence of λ on h.

There is an 'ordinary series' relating the separable cylindrical solutions $\exp(ihz)J_1(\lambda\rho)$ to the separable spherical solutions $j_n(Kr)P_n^1(\cos\theta)$ of the conventional wave equation.

Each term of the series (5.4), (5.5) and (5.6) has the same dependence on θ as a term of the corresponding external solutions. This has enabled us to equate coefficients of the θ terms at the boundary and so render the resonance condition (5.11) θ independent.

The equation (5.11) appears to be a new type of eigenequation, whose solution has yet to be obtained in a rigorous way.

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Appendix. Estimate of error

The results for the resonance problem presented in § 6 are based on the assumption that

$$\int_0^1 dx\, F(x) j_n\left(\frac{y}{x}\right) \sim j_n(y) \int_0^1 dx\, F(x) \tag{A1}$$

where $F(x)$ is a smooth function compared with $j_n(y/x)$. This is a drastic assumption which, however, makes the problem soluble and may provide starting values for a numerical attack on the problem. (We do *not* assume that $j_n(y/x) \sim j_n(y)$ for all x in the interval 0 to 1.)

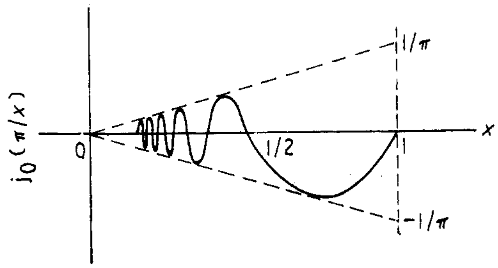

Figure 5.

Let us consider the behaviour of the functions $j_n(y/x)$ in the range (0 to 1). For example

$$j_0\left(\frac{\pi}{x}\right) = \frac{x}{\pi} \sin\left(\frac{\pi}{x}\right).$$

The functions $j_n(y/x)$ oscillate with infinite frequency as $x \to 0$. The integral

$$\int_0^1 dx\, g(x) j_n\left(\frac{y}{x}\right)$$

where $g(x)$ is a comparatively smooth function, oscillates in a similar manner, so that values of x near zero contribute little to the integral. The value of the integral is determined for the most part by the behaviour of $j_n(y/x)$ near $x = 1$. This suggests the approximation (A1), in which $j_n(y)$ is the first term of a Taylor expansion of $j_n(y/x)$ about $x = 1$.

It is possible to obtain an estimate of the error by the arguments which follow, but even here small values of x in the integrand of (A1) are to be excluded.

We have obtained approximate solutions $f_n{}^0(x)$ and $y^0{}_{,s}$ of the simultaneous equations

$$\int_{-1}^1 dx\, g_n(x, y) f(x) = 0 \tag{A2}$$

where

$$g_n(x, y) = P_n{}^1(x^2) j_{n-1}\left(\frac{y}{x}\right).$$

Substitution of any one of the approximate solutions in (A2) yields a set of non-zero constants on the left which are to be compared with zero on the right. This is a meaningless exercise; the method of substitution fails.

However, the error in the approximate solutions can be estimated. In particular, for the principal mode we have

$$f_1{}^0(x) = |x| P_1{}^1(x^2) \quad \text{and} \quad y_1{}^0 = \pi.$$

We seek improved solutions of the form

$$f_1(x) = f_1{}^0(x) + \delta_1 E_1(x)$$

$$y_1 = y_1{}^0 + \delta_1$$

where

$$|\delta_1| \ll |y_1{}^0|.$$

Now

$$g_n(x, y_1) = g_n(x, y_1^0+\delta_1)$$
$$\sim g_n(x, y_1^0)+\delta_1 g_{n,2}(x, y_1^0)$$

where

$$g_{n,2}(x, y) = \frac{\partial}{\partial y} g_n(x, y).$$

We have to solve now

$$\delta_1 \int_{-1}^{1} \mathrm{d}x\, G_n(x)E_1(x) = -A_n - \delta_1 B_n \tag{A3}$$

with

$$G_n(x) = g_n(x, y_1^0), \qquad \text{a function of } x \text{ alone;}$$
$$A_n = \int_{-1}^{1} \mathrm{d}x\, g_n(x, y_1^0) f_1^0(x), \qquad \text{a constant;}$$

and

$$B_n = \int_{-1}^{1} \mathrm{d}x\, g_{n,2}(x, y_1^0) f_1^0(x), \qquad \text{a constant.}$$

We consider the expansion

$$g_n(x, y_1^0) = g_n^0(x, y_1^0)+\delta_1 h_n(x, y_1^0) \tag{A4}$$

where $g_n^0(x, y_1^0)$ is our approximation to g_n resulting from the assumption (A1). That is

$$g_n^0(x, y_1^0) = L_n(y_1^0)\chi_n(x)$$

with

$$L_n(y) = \mathrm{j}_{n-1}(y)$$

and

$$\begin{aligned} \chi(x) &= \mathrm{P}_n^1(x^2), && n \text{ odd} \\ &= -\mathrm{P}_n^1(x^2), && n \text{ even and } -1 \leqslant x \leqslant 0 \\ &= \mathrm{P}_n^1(x^2), && n \text{ even and } 0 \leqslant x \leqslant 1. \end{aligned}$$

The $\chi_n(x)$ are orthogonal with respect to the density function x and

$$\begin{aligned} \int_{-1}^{1} \mathrm{d}x\, \chi_n(x)\chi_m(x)x &= 0, && n \neq m \\ &= \frac{m(m+1)}{2m+1}, && n = m \end{aligned}$$

but $L_1(y_1^0) = 0$, so that $g_1^0(x, y_1^0) = 0$. The correction term $\delta_1 h_1$ is just g_1, so that the expansion (A4) is not permissible for $n = 1$. We retain it for $n > 1$. Equations (A3) are now

$$\delta_1 \int_{-1}^{1} \mathrm{d}x\, G_1(x)E_1(x) = -A_1 - \delta_1 B_1 \tag{A5}$$

and

$$\delta_1 L_n(y_1^0) \int_{-1}^{1} \mathrm{d}x\, \chi_n(x)E_1(x) = -A_n - \delta_1 B_n, \qquad n = 2 \dots \infty. \tag{A6}$$

The second-order terms in δ_1 have been neglected. We now assume that $E_1(x)$ can be expanded as

$$E_1(x) = \sum_{m=2}^{\infty} e_m f_m^0(x). \tag{A7}$$

Since $f_m{}^0(x)$ are orthogonal to $\chi_n(x)$, equations (A6) reduce to

$$\delta_1 L_n(y_1{}^0)\frac{n(n+1)}{2n+1}e_n = -A_n - \delta_n B_n$$

so that

$$e_n = -\frac{(A_n+\delta_1 B_n)(2n+1)}{\delta_1 L_n n(n+1)}$$

and $E_1(x)$ is determined as a function of δ_1. Equation (A6) can be used now to determine δ_1. If we assume (A7) to converge rapidly, we obtain for $E_1(x)$, by taking only the first term in the series,

$$E_1(x) \sim \frac{A_2+\delta_1 B_2(4+1)}{\delta_1 L_2{}^2(2+1)} f_2{}^0(x).$$

This gives

$$\delta_1 \sim \frac{A_1-D}{C-B_1}$$

where

$$C = \frac{5B_2 I}{6L_2}$$

and

$$D = \frac{5A_2 I}{6L_2}$$

and

$$I = \int_{-1}^{1} \mathrm{d}x\, G_1(x) f_2{}^0(x).$$

The integrals are calculated numerically and give the results

$$A_1 = -0{\cdot}036, \qquad B_1 = 0{\cdot}0090, \qquad A_2 = -0{\cdot}0031$$
$$B_2 = 0{\cdot}061, \qquad I = 0{\cdot}38, \qquad I_2(y_1{}^0) = \mathrm{j}_1(\pi) = 0{\cdot}318$$

whence $C = 0{\cdot}060$, $D = -0{\cdot}0031$ and $\delta_1 = -0{\cdot}65$. This represents an error of about 20% in $y_1{}^0$.

Now $\delta_1 E_1(x) = 0{\cdot}11 f_2{}^0(x)$, giving an error of about 10% in the solution $f_1{}^0(x)$.

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